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SUMMARY
of Dainis Zeps doctoral thesis: The theory of combinatorial maps and its use in the graph-topological computations

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Thesis and its summary is available at the library of the University of Latvia,
Kalpaka bulvāris 4.

General information

This work is elaborated basing on the investigations of the last years about graph rotational schemes, but in the same time it is as a summary of the longer work, that started in the leadership of Emanuel Grinberg in 1978. Solving graph theoretical problems under the leadership of Emanuel Grinberg and trying to find for them the applications in the real life, we came to the ideas and methods, that we put further in the basis of the construction of effective algorithms. The first such algorithmical solutions were connected with graph planarity and threeconnectivity, which according their nature pertain to the graph topology. This topological direction has been kept all these years of investigations, and therefore it is a logical outcome of the investigations to arrive to the central subject of this work - graph rotational schemes.

The investigations of previous years in graph theoretical questions and in the construction of graph theoretical algorithms always have gone hand in hand, what is reflected also in the results of investigations, i. e. in our contribution what we have given in this area. In this way the practical task about the implementation of the nonplanar graph in the plane by the implementation of the intersections of the edges gave an impulse to investigations about dynamical partition of the graph into threeconnected components. The results what was achieved are in the work [7]. Unfortunately the work was not published further than republican fund of algorithms. With the year 1990 beginning there appeared a whole sequence of works [for example [38, 39, 40]], which also were solving the same problem, the solution of which we have locally published already in the year 1984.

The difficulty of the problem gave an impulse for theoretical investigations about the threeconnectivity of the graph in dynamical consideration, where the graph has been considered as a sequence of the edges. The Tutte's theory classical in this dynamical case turned out to be not valid sufficiently. An alternative theory which was directed to dynamical partition of the graph into threeconnected components were created and the results were reflected in the work [9].

In this theory the graph was considered in the beginning as being empty and stepwise the new edges were added to it, in each step the graph being maintained partitioned into threeconnected components. Controversially to the Tutte's theory where the graph was considered as a set of edges [correspondingly also the components of three types – threeconnected graphs, polygons and bonds with the added virtual edges], the graph in our dynamical theory was built from bases which are subsets of vertices of the graph already partitioned into threeconnected components, where a type of the basis corresponds to the type of a threeconnected component in the Tutte's theory.

In the same time, when the theory of the dynamical partition of the graph into threecomponents were developed, the program on computer was implemented, that maintained the partition of the graph into threecomponents dynamically when new edges were added to the graph. The program was very complicate what in the turn forced to reconsider many of our conceptions in the construction of algorithms that were formed during previous years. All these facts facilitated to the formation of a new theory.

When the computer program was under construction, we decided that besides the threeconnected components must also be maintained the dual graphs of the threeconnected components. There arose a question what must be considered as a dual graph of the nonplanar graph. The question was solved by discovering the simplest schema of rotation of the graph. Further it turned out that this schema is well known fact, which has been rediscovered by many of the mathematicians. As the first is acknowledged the German mathematician of the 19th century Hefter [21], who knew this schema already in his time. More widely the discoverer and the author of the idea that fixing the order of the edges around the vertex fixes the embedding of the graph on some surface, has been considered the mathematician Edmonds. After we have come to this point, the investigations started about the schemata of graph rotations. Today this subject is called the theory of combinatorial maps.

In 1993 we came to the thought to use in the duality schema in the place of the vertices and the edges other objects, namely corners between next laying edges in the fixed rotation of the edges around vertices. In this case the duality schema itself becomes selfdual, because the dual object of the corner is again a corner. Investigating these schemata we came directly to combinatorial maps. Working three moths in the Prague university in the group of the professor Nešetřil this theory was developed to some ripening and the results are reflected in the work [12].

Today the theory of combinatorial maps is rapidly developing, what can be seen from the enormous amount of works in this area and that the first book on the subject has appeared, the monograph written by in the New Zealand working mathematicians Bonnington un Little 'Foundations of topological graph theory' [16].

The main theme of this dissertation is the theory of combinatorial maps, and the main results have been reflected in three published works [12, 13, 15].

Graphs itself are very simple objects, but the embedding of the graph on combinatorial surface, as it turns out, is even more simple combinatorial object. Combinatorial maps as combinatorial objects can be introduced in several ways, for example

1) permutational pairs with the condition that their multiplication is an involution without fixed points [graphs on orientable surfaces];
2) permutational pairs without additional condition [multigraphs on orientable surfaces];
3) 3-graphs [16] [multigraphs on nonorientable surfaces];
4) triples of involutions without fixed elements [multigraphs on nonorientable surfaces];
5) Tutte's defined combinatorial maps [33][graphs on nonorientable surfaces].

Our investigation touches all these representations, but the main attention is turned to the first two of these representations. In our approach there are two different conceptions in the interpretation of these objects. Firstly, as elements upon which permutations act we consider purely geometrical objects, namely, corners between edges. Secondly, in the place of multigraphs we consider partial graphs, i.e. subgraphs.

The investigations of this work comprise the maps itself, their features, the classes of maps partitioning them due to the combinatorial map of the map, their isomorphism. Investigating the partial maps the notion of their image has been introduced. The image is easy calculable using permutational calculus, and this technique turns out to be very effective in some subcases, when the coloring into two colours of the elements is given. Further the theory of colorable maps has been developed, which gives interesting geometrical objects on surfaces, the cycles of corners in the embeddings of graphs, which in it turn gives the partition of the edges into cycle edges and cut edges.

One of the principal conceptions of this work is that we establish one-one map between permutations and graphs on surfaces. Normalising one of the edge rotations of the combinatorial map we get that the map is determined by only one permutation, i.e. in the class of all maps for each map, namely permutation there corresponds one graph on the surface. Widening this conception its possible to say that for each permutational formula there corresponds some geometrical object on the surface, that is calculable with this formula.

This has been developed organising computations in the computer environment. During the work it turned out that many necessary objects for the graphs on surfaces can be computed using permutations and permutational formulas, where the simplest operations with permutations are used, namely, the multiplication of permutations and the restriction of permutation on some specified subset.

When some simplest algorithms have been added, for example the known search zizag-walk, that gives the combinatorial knot of the map, the linear search of the cycles of the map, then the possible array of the computations widens. Nevertheless it is not sufficient this set of operations in order to perform some serious algorithmic solution, for example, graphs planarisation or dynamic maintenance of the graphs threecomponents.

Nevertheless just this direction is that where we see the main motivation of our work, because the more deep investigation of combinatorial maps, characteristic objects and their computability using simplest operations gives a hope, that the set of simples operations must grow, and we do not see any principal obstacles, that these calculations couldn't comprise also more complex algorithms.

The thesis has 61 pages and 15 chapters, the main results are published in the papers [12, 14, 15].

The summary of the thesis through chapters

The first chapter is preface that has general notices about the nature of the work.

The second chapter is introduction that consists of four parts called preliminary considerations. The first preliminary considerations looks on rotational schemata and our approach in their treatment. With the prehistory beginning the main conceptional approach has been explained in this work concerning the one-one map between permutations and graphs on surfaces and how this mapping is used further in the thesis. The second preliminary considerations speaks about permutational computations and the algorithmical environment on computer. Three types of operations are defined, correspondingly these that can be done in the frames of permutations, the simples algorithmic operations expressible in permutations and all other operations.

The forth preliminary considerations touches principal questions what means to use the theory of combinatorial maps in graph topological computations.

The third chapter consider permutations in general and the denotations used in the work.

The forth chapter gives a short notice where to find some more wide considerations of combinatorial maps that used in this work.

The fifth chapter with its six subchapters consider combinatorial maps that are defined as pairs of permutations with one additional condition, that the edge rotation contain only graphical edges. In this chapter some simples features of the maps, the closed classes of maps against multiplication with fixed right edge rotation and the normalised maps are introduced.

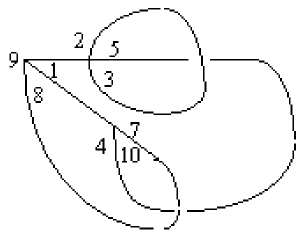
Example 1. *The example of the combinatorial map:*

$$P = (189)(2536)(47\bar{0})$$

$$Q = (17926)(3548\bar{0})$$

$$\pi = (12)(34)(56)(78)(9\bar{0})$$

$$\varrho = (14)(23)(5\bar{0})(69)(78)$$



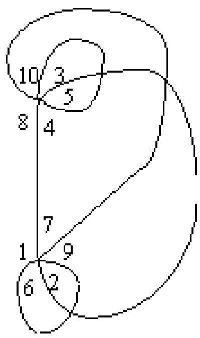
Picture 1: The drawing of the map in the example

In the third subchapter the classes of maps with fixed left edge rotation has been considered. It is proved that the class of selfconjunged maps comprise a class of maps where the right and left rotations coincide, and this class is a subgroup of the group of maps but other classes with fixed left edge rotation are the cosets of this subgroup.

The forth subchapter consider the knot of the combinatorial map and shows that for every class with fixed left edge rotation the combinatorial knot is fixed. The theorem is proved that every map can be expressed as the multiplication of the knot of this map with its selfconjunged map that is named knotting.

Example 2. *K_4 corresponding normalised combinatorial map:*

$$P = (19\bar{1})(4\bar{2}8)(236)(57\bar{0})$$



Picture 2: The drawing of the dual map in the example

$$Q = (1\bar{0}6)(24\bar{1})(358)(79\bar{2})$$

$$\varrho = (17)(28)(3\bar{0})(49)(5\bar{2})(6\bar{1})$$

The knot of this map is:

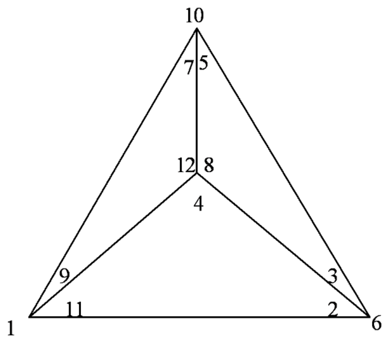
$$\mu = (1287)(349\bar{0})(56\bar{1}\bar{2})$$

The corresponding knotting is:

$$\alpha = (1\bar{0}\bar{1}29\bar{2})(358)(467)$$

In the next subchapters the isomorphism and graphical isomorphism of maps have been considered.

The sixth chapter consider partial combinatorial maps, which turns out to be very useful notion in the map investigation.



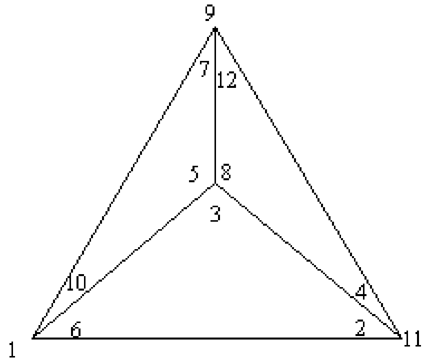
Picture 3: The drawing of the map considered in the example

This chapter contains 11 subchapters. The first subchapter consider drawing of partial maps, where the procedure of the drawing in the same time is a proof that for an arbitrary pair of permutations corresponds graphical partial map, that is, a submap of some map. This conception is formalised introducing combinatorial object – the image of the partial map that itself is a combinatorial map.

Example 3. *Partial map*

$$\left\{\begin{array}{l}(12)(34)(56)\\ \underline{(135)(246)}\\ (145236)\end{array}\right.$$

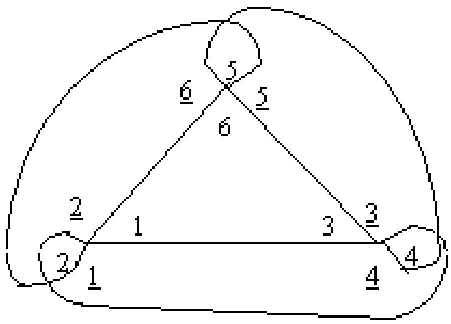
The image of this partial map:



Picture 4: The drawing of the dual map considered in the example

$$\left\{\begin{array}{l} (1\bar{1}2\bar{2})(3\bar{3}4\bar{4})(5\bar{5}) \\ (135)(246)(\bar{6}\bar{3}\bar{2}\bar{5}\bar{4}\bar{1}) \\ \hline (1\bar{4})(2\bar{3})(3\bar{6})(4\bar{5})(5\bar{2})(6\bar{1}) \end{array}\right.$$

In the third chapter the geometrical interpretation of the nongraphical edges of partial map has been considered. This interpretation gives possibility to consider the partial map as a submap of such a map, which is received by glowing up the nongraphical edges [considered as empty] with polygons. This map is called reduced [or shortened] image of the partial map.



Picture 5: The drawing of the image of the partial map

Subchapters from 6 to 11 consider some more specific questions.

The seventh chapter has only short notice about literature where can be found the treatment of the graphs on nonorientable surfaces.

The eighth chapter considers theory of cycle covers. The cycle cover is called such a permutation that's application is a choice between vertex and face rotations. The orbits in the cycle cover geometrically interpreted are cycles in the graph, at what for every cycle in the graph there exists such a cycle cover that contains the corresponding orbit. Though a sample of the cycle covers contains only a small part of all the cycles of the graph, nevertheless these objects are very simple and their investigation as topological objects in the place of all the set of cycles of the graph gives considerable results. Because of this an important role has been bestowed to the investigation of these objects. Comparing with the well known in literature Stahl's cycles [30], which are useful by proving the Jordan curve theorem in combinatorial outset, but they are too hard to work with them.

The cycle cover partitions the edges of the map into four types, correspondingly cycle-, cut-, cross- and recurrence-edges.

Example 4. *Normalised planar combinatorial map, that corresponds to the prism graph:*

$$P = (1\bar{3}7)(2\bar{0}\bar{1})(38\bar{6})(4\bar{7}9)(5\bar{5}\bar{4})(6\bar{2}\bar{8})$$

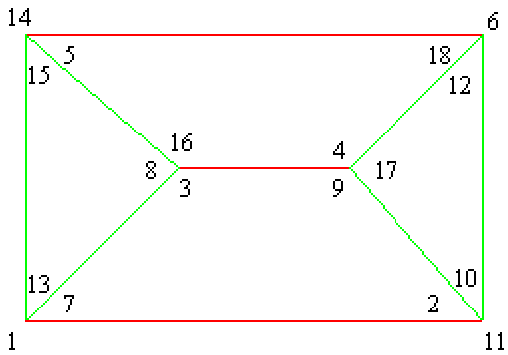
$$Q = (1\bar{4}6\bar{1})(2937)(4\bar{8}5\bar{6})(8\bar{5}\bar{3})(\bar{0}\bar{2}\bar{7}).$$

Cycle cover without inner edges [in this case the only possible]:

$$(1\bar{4}5\bar{6}37)(294\bar{8}6\bar{1})(8\bar{5}\bar{3})(\bar{0}\bar{2}\bar{7}).$$

Taking only cyclical edges, i.e. cycle- and recurrence-edges, and restricting on them the cycle cover, we get the so called cycle cover submap. The theorem is proved, that the cycle cover submap is (graphical) combinatorial map.

An important point is that cycle covers and their characteristics is very easy computable objects. The expressions that contain cycle covers and partition of the edge rotation into edge types are considered in the eighth subchapter. The tenth subchapter consider important theorems, that consider the changes in the edge types when it is multiplied with an edge from the right edge rotation.



Picture 6: The drawing of the prism graph with marked cycle-edges.

The ninth chapter considers an important subcase of cycle covers when only edges of two types, namely – cycle- and cut-edges, are present. In this case the set of the elements of the map is colorable into two colours in the way that every edge in the edge rotation receives both colours. These cycle covers are called colorable cycle covers. This case is important because of the fact that cycle covers are computable with the map search algorithm that is called zigzag-walk. As it is known the zigzag-walk fixes the combinatorial knot of the map.

In 9.2 some results of the application of colorable cycle covers in graph topological computations have been considered. In 9.2.2 and 9.2.3 an approach is demonstrated that is formulated in the introduction: proving an theorem[theorem 55] which has a distinct graphical interpretation, its consequences in graph topological outlook has been considered. By the way a simple routine how to cut cycles in the colorable cycle cover is shown.

Tenth chapter considers a technique of permutational calculus in the case when the set of elements is partitioned into subsets.

11th chapter this technique is used in order to calculate the image of the partial map and its characteristics. The knot of the map and knotting are calculated. Further this technique is used in order to calculate the partial maps of both colours defined by the colorable cycle cover.

12th chapter considers some graph topological operations how they are to be implemented with intertwined computations of partial maps. The types of operations have been investigated via their complexity, using a simple classification of these operations.

In 13th chapter is the environment for computation of combinatorial maps and their characteristics implemented in PASCAL considered. This environment is used to test different algorithms and hypotheses. The maps are entered both manually and generated randomly. The size of the graphs are several hundred edges. In the page 54 is given a protocol of an experiment in this environment.

14th chapter considers the prehistory of these investigations, that are reflected in this thesis.

15th chapter is the summary of the work, where the question of our contribution in the graph theoretical and algorithmical problems has been touched.

16th chapter contains acknowledgements.

In the end of the work the references to literature and other kind of notices[reviews in conferences, nonpublished manuscripts] are given, all together 62 items, partitioned into three groups. The first group [26 items] contains references to the works of other authors, that have some connection with the rotational schemata. Second group [6 items] contains references to important works that have connection with work of the previous years in the field of the construction of graph theoretical algorithms. Third group [30 items] contains references to the works [oral in conferences and nonpublished manuscripts too] of the author both of the previous years and also of the latest years that have direct connection with the main theme of this thesis.

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